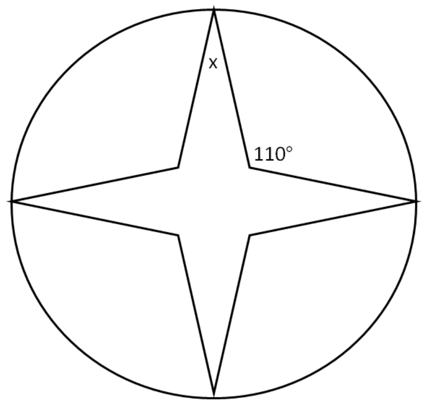
IMO Geometry Questions

Level: Intermediate   Ref No: M01   Puzz Points: 10

[Cayley 2004 Q1] The “star” octagon shown in the diagram is beautifully symmetrical and the centre of the star is at the centre of the circle. If angle NAE = 110°, how big is the angle DNA?



Solution: 20°

Level: Intermediate   Ref No: M03   Puzz Points: 10

[Cayley 2004 Q3] A quadrilateral ABCD has sides AB, BC, CD, DA of length x, y, z and t, respectively. The diagonals AC and BD cross at right angles. Prove that:

$$x^2 + z^2 = y^2 + t^2$$

Level: Intermediate   Ref No: M08   Puzz Points: 15

[Hamilton 2004 Q2] Triangle ABG has a right angle at B. Points C and E lie on side AG and points D and F lie on side BG so that the six line segments AB, BC, CD, DE, EF and FG are equal in length. Calculate the angle AGB.

Solution: 15°

Level: Intermediate Ref No: M12 Puzz Points: 18

[Hamilton 2004 Q6] The triangle ABC is right-angled at A, with AB = 6cm and AC = 8cm. Points X and Y are situated on BC such that AB = AY and AX = XC. Two isosceles triangles ABY and AXC are thus created. These triangles overlap, forming the region AXY. Calculate the area of this region.

Solution: $\frac{132}{25}$

Level: Intermediate Ref No: M13 Puzz Points: 20

[Maclaurin 2004 Q1] A quadrilateral is enclosed by four straight lines with equations:

$$2y = x + 4$$
$$y = 2x - 4$$
$$2y = x - 2$$
$$y = 2x + 2$$

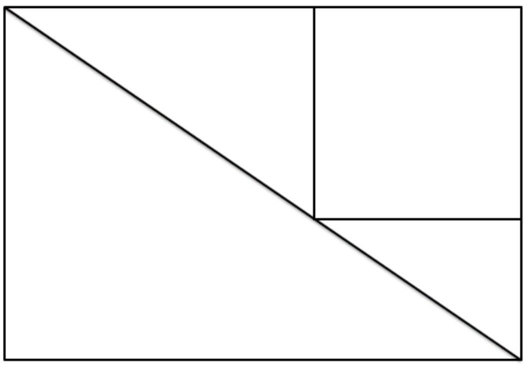
Calculate the area of this quadrilateral.

Solution: 12

Level: Intermediate Ref No: M15 Puzz Points: 20

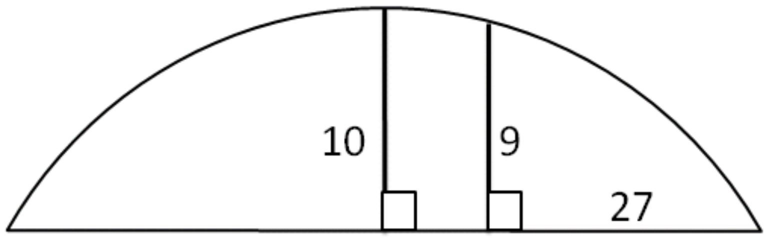
[Maclaurin 2004 Q3] A square is constructed inside a rectangle of length *a* and width *b*, with the square touching the diagonal of the rectangle as shown in the diagram. If the square has side *h*, prove that:

$$\frac{1}{h} = \frac{1}{a} + \frac{1}{b}$$



Level: Intermediate   Ref No: M18   Puzz Points: 23

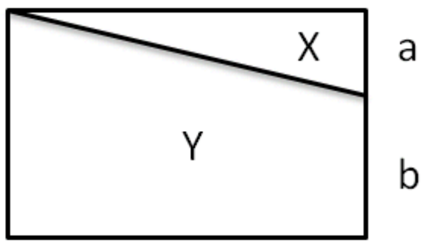
[Maclaurin 2004 Q6] The cross section of a tunnel is a circular arc, as shown in the diagram. The maximum height of the tunnel is 10 feet. A vertical strut 9 feet high supports the roof of the tunnel from a point 27 feet along the ground from the side. Calculate the width of the tunnel at ground level.



Solution: 80 feet

Level: Intermediate   Ref No: M19   Puzz Points: 10

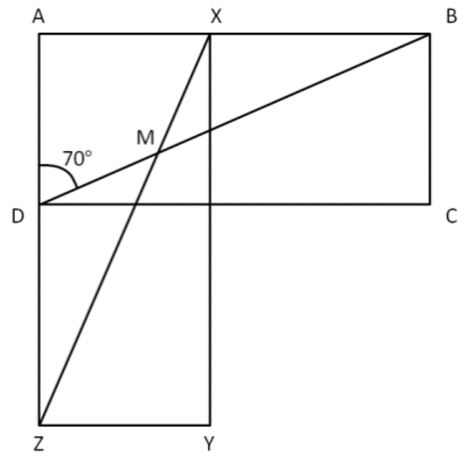
[Cayley 2006 Q1] A rectangular piece of paper is cut into two pieces by a straight line passing through one corner, as shown. Given that area X : area Y = 2:7, what is the value of the ratio a:b?



Solution: 4:5

Level: Intermediate Ref No: M21 Puzz Points: 10

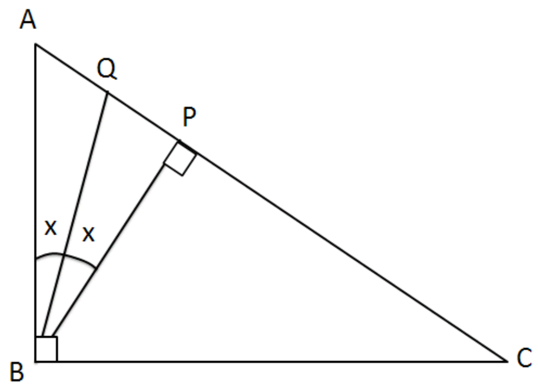
[Cayley 2006 Q3] In the diagram, rectangles ABCD and AZYX are congruent, and $\angle ADB = 70°$. Find $\angle BMX$.



Solution: $50°$

Level: Intermediate Ref No: M26 Puzz Points: 15

[Hamilton 2006 Q2] In triangle ABC, $\angle ABC$ is a right angle. Points P and Q lie on AC; BP is perpendicular to AC; BQ bisects $\angle ABP$. Prove that CB = CQ.

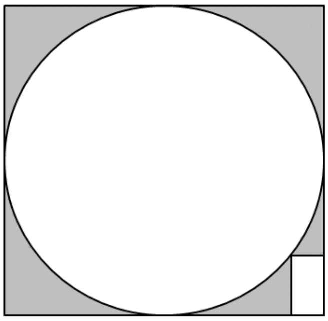


Solution: Just need to show that $\angle QBC = \angle BQC$, i.e. we have an isosceles triangle.

Level: Intermediate   Ref No: M28   Puzz Points: 15

[Hamilton 2006 Q4] A circle is inscribed in a square and a rectangle is placed inside the square but outside the circle. Two sides of the rectangle lie along sides of the square and one vertex lies on the circle, as shown. The rectangle is twice as high as it is wide.

What is the ratio of the area of the square to the area of the rectangle?

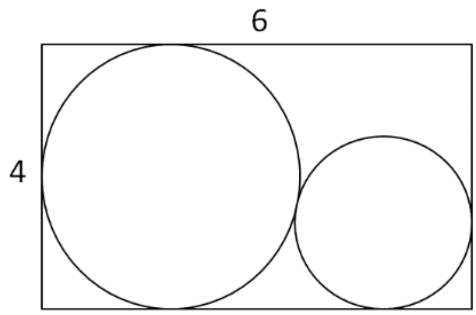


Solution: 50:1

Level: Intermediate   Ref No: M33   Puzz Points: 20

[Maclaurin 2006 Q3] Two circles are drawn in a rectangle of 6 by 4, such that the larger circle touches three sides of the rectangle, whereas the smaller one only touches 2. Determine the radius of the smaller circle.

Solution: $8 - 4\sqrt{3}$

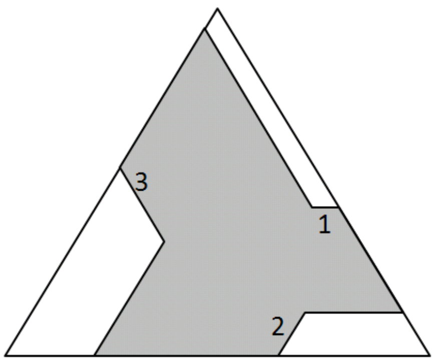


Level: Intermediate   Ref No: M34   Puzz Points: 20

[Maclaurin 2006 Q4] The nonagon shown shaded in the diagram has been made by removing three pieces from an equilateral triangle of side 12. All nine edges of the nonagon are parallel to sides of the triangle. Three edges have lengths 1, 2 and 3 as shown.

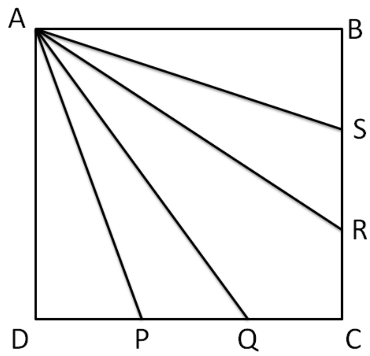
Calculate the length of the perimeter of the nonagon.

Solution: 30



Level: Intermediate Ref No: M39 Puzz Points: 10

[Cayley 2007 Q3] The diagram shows a square ABCD of side 10 units. Line segments AP, AQ, AR and AS divide the square into five regions of equal area, as shown. Calculate the length of QR.



Solution: $\sqrt{8} = 2\sqrt{2}$ units.

Level: Intermediate Ref No: M40 Puzz Points: 10

[Cayley 2007 Q4] How many right-angled triangles can be made by joining three vertices of a cube?

Solution: 48

Level: Intermediate Ref No: M41 Puzz Points: 13

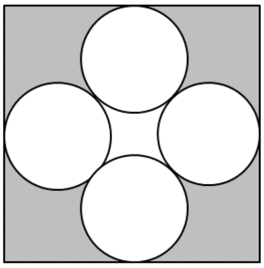
[Cayley 2007 Q5] In a quadrilateral ABCD, AB = BC, $\angle BAC = 60°$, $\angle CAD = 40°$, AC and BD cross at X and $\angle BXC = 100°$.

Calculate $\angle BDC$.

Solution: $30°$

Level: Intermediate Ref No: M45 Puzz Points: 15

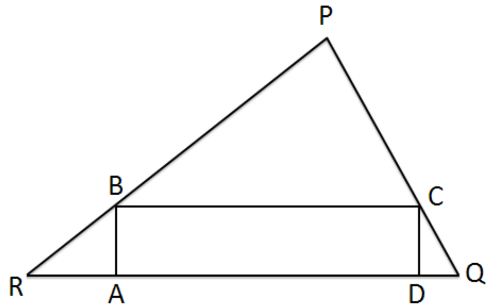
[Hamilton 2007 Q3] The diagram shows four circles of radius 1 placed inside a square so that they are tangential to the sides of the square at the midpoints of the sides, and to each other. Calculate the shaded area.



Solution: $8(1+\sqrt{2}) - 3\pi$

Level: Intermediate   Ref No: M47   Puzz Points: 18

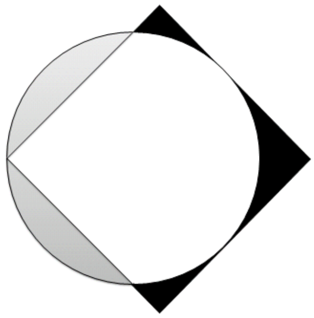
[Hamilton 2007 Q5] The diagram shows a rectangle ABCD inscribed inside a triangle PQR. The side, AB, of the rectangle is one third of the perpendicular height of the triangle from P to QR. What is the ratio of the area of the rectangle to the area of the triangle?



Solution: $4:9$

Level: Intermediate   Ref No: M50   Puzz Points: 20

[Maclaurin 2007 Q2] The diagram shows a circle of radius 2 and a square. The circle touches two sides of the square and passes through one corner of the square. The area of the region shaded black (inside the square but outside the circle) is X and the area of the region shaded grey (inside the circle but outside the square) is Y.



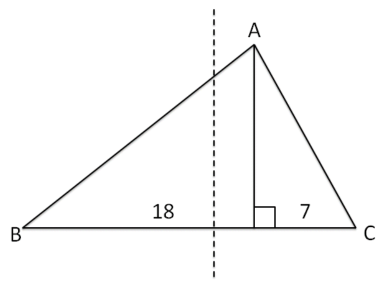
What is the value of $Y - X$?

Solution: $4\pi - 6 - 4\sqrt{2}$

Level: Intermediate   Ref No: M52   Puzz Points: 20

[Maclaurin 2007 Q4] The diagram shows a triangle in which the altitude from A divides the base, BC, in the ratio 18 : 7.

Find the ratio in which the base is divided by a line parallel to the altitude which cuts the triangle into two equal areas.



Solution: $3:2$

Level: Intermediate   Ref No: M53   Puzz Points: 23

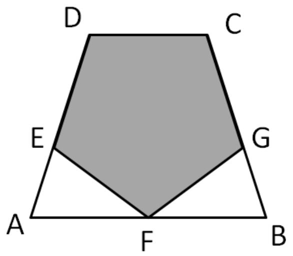
[Maclaurin 2007 Q5] The coordinates of three vertices of a cube are (4, 0, 3), (6, 4, 1) and (2, 8, 5). Find the coordinates of a fourth vertex.

Solution: (0, 4, 7)

Level: Intermediate Ref No: M56 Puzz Points: 10

The diagram shows a regular pentagon $CDEFG$ inside a trapezium $ABCD$.

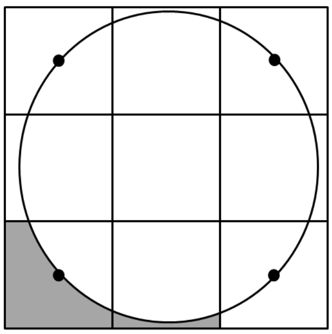
Prove that $AB = 2 \times CD$.



Level: Intermediate Ref No: M58 Puzz Points: 10

[Cayley 2011 Q4] The diagram shows nine 1cm × 1cm squares and a circle. The circle passes through the centres of the four corner squares.

What is the area of the shaded region – inside two squares but outside the circle?

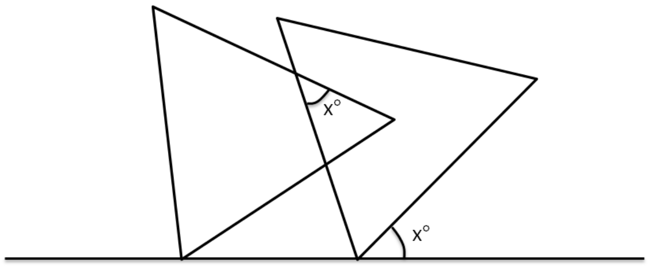


Solution: $\frac{9-2\pi}{4}$

Level: Intermediate Ref No: M62 Puzz Points: 15

[Hamilton 2011 Q2] The diagram shows two equilateral triangles. The angle marked $x°$ are equal.

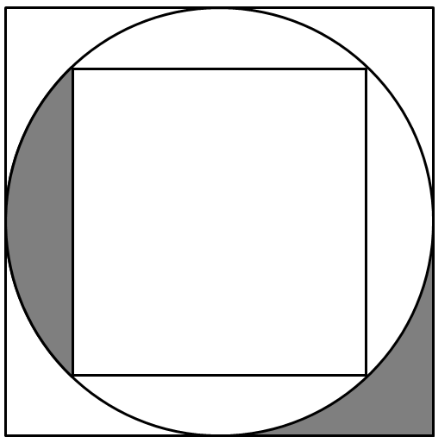
Prove that $x > 30$.



Solution: Angle at bottom of left triangle, on the right side of the point, is $2x - 60$. Since $2x - 60 > 0$, then $x > 30$.

Level: Intermediate Ref No: M64 Puzz Points: 15

[Hamilton 2011 Q4] A square just fits within a circle, which itself just fits within another square, as shown in the diagram. Find the ratio of the two shaded areas.

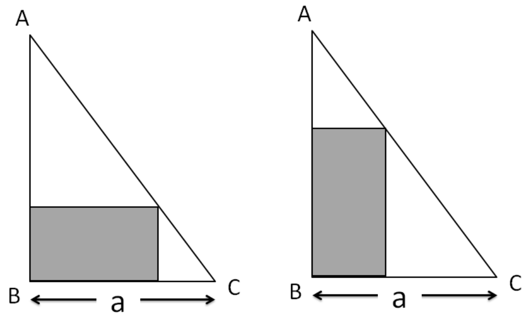


Solution: $(4 - \pi): (\pi - 2)$

Level: Intermediate Ref No: M69 Puzz Points: 20

[Maclaurin 2011 Q3] The diagrams show a rectangle that just fits inside right-angled triangle *ABC* in two different ways. One side of the triangle has length *a*.

Prove that the perimeter of the rectangle is *2a*.

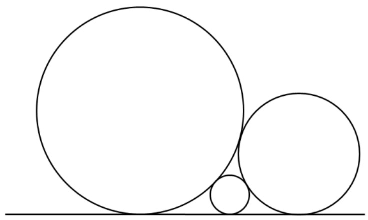


Level: Intermediate Ref No: M71 Puzz Points: 23

Three circles touch the same straight line and touch each other, as shown.

Prove that the radii *a*, *b*, and *c*, where *c* is smallest, satisfy the equation:

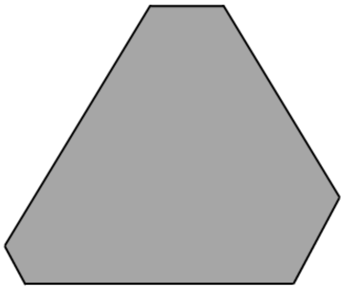
$$\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}=\frac{1}{\sqrt{c}}$$



Level: Intermediate Ref No: M74 Puzz Points: 10

[Cayley 2008 Q2] A hexagon is made by cutting a small equilateral triangle from each corner of a larger equilateral triangle. The sides of the smaller triangles have lengths 1, 2 and 3 units. The lengths of the perimeters of the hexagon and the original triangle are in the ratio 5:7.

What fraction of the area of the original triangle remains?



Solution: $\frac{5}{7}$

Level: Intermediate Ref No: M75 Puzz Points: 10

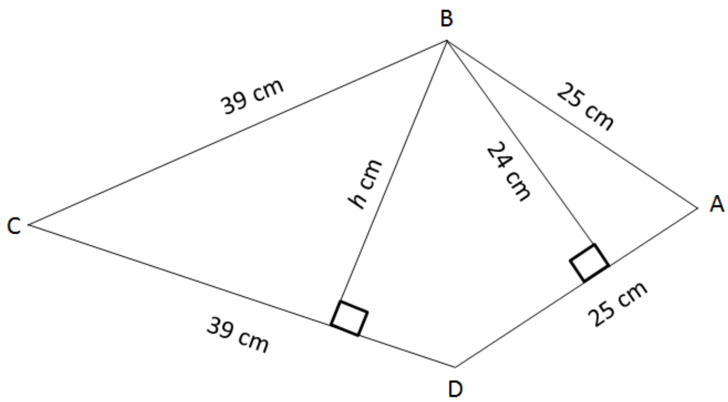
[Cayley 2008 Q3] In the rectangle ABCD the midpoint of AB is M and AB : AD = 2 : 1. The point X is such that triangle MDX is equilateral, with X and A lying on opposite sides of the line MD. Find the value of $\angle XCD$.

Solution: $30°$

Level: Intermediate Ref No: M77 Puzz Points: 13

[Cayley 2008 Q5] A kite has sides AB and AD of length 25cm and sides CB and CD of length 39cm. The perpendicular distance from B to AD is 24cm. The perpendicular distance from B to CD is *h* cm.

Find the value of *h*.



Solution: $h = \frac{360}{13}$

Level: Intermediate Ref No: M78 Puzz Points: 13

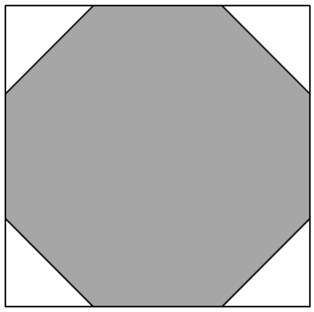
[Cayley 2008 Q6] A regular tetrahedron ABCD has edges of length 2 units. The midpoint of the edge AB is M and the midpoint of the edge CD is N.
Find the exact length of the segment MN.

Solution: $\sqrt{2}$

Level: Intermediate Ref No: M79 Puzz Points: 15

A regular octagon with sides of length a is inscribed in a square with sides of length 1, as shown.

Prove that $a^2 + 2a = 1$.



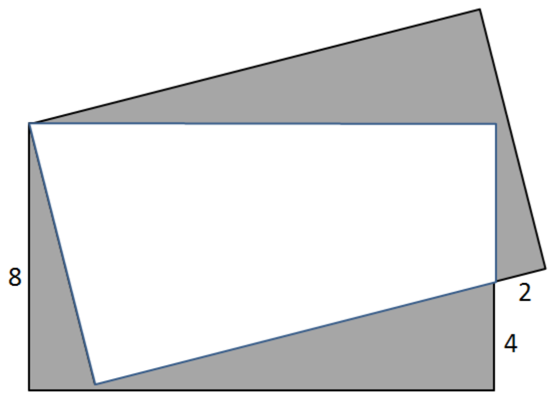
Level: Intermediate Ref No: M81 Puzz Points: 15

[Hamilton 2008 Q4] A triangle is bounded by the lines whose equations are
$y = -x - 1$, $y = 2x - 1$ and $y = k$, where k is a positive integer.
For what values of k is the area of the triangle less than 2008?

Solution: $1 \leq k \leq 50$

Level: Intermediate Ref No: M82 Puzz Points: 18

[Hamilton 2008 Q5] Two congruent rectangles have a common vertex and overlap as shown in the diagram. What is the total shaded area?

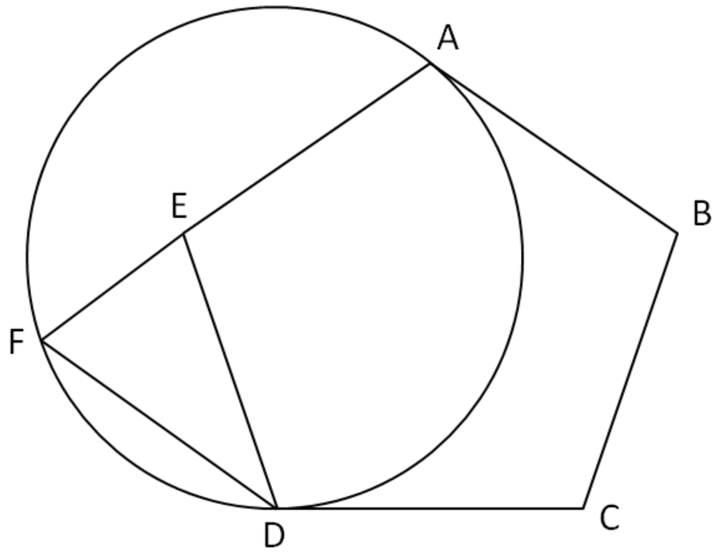


Solution: 68

Level: Intermediate   Ref No: M85   Puzz Points: 20

[Maclaurin 2008 Q2] The diagram shows a regular pentagon ABCDE. A circle is drawn such that AB is a tangent to the circle at A and CD is a tangent to the circle at D. The side AE of the pentagon is extended to meet the circumference of the circle at F.

Prove that DE = DF.

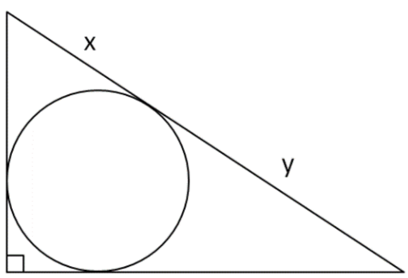


Solution: Since CD and AB are tangents to the circle, it would seem sensible to add radii connecting A and D to the centre of the circle. Appropriate use of circle theorems and interior angles of regular polygon allows us to eventually show that $\angle EFD = \angle FED$.

Level: Intermediate   Ref No: M87   Puzz Points: 20

[Maclaurin 2008 Q4] A circle is inscribed in a right-angled triangle, as shown. The point of contact of the circle and the hypotenuse divides the hypotenuse into lengths x and y.

Prove that the area of the triangle is equal to xy.

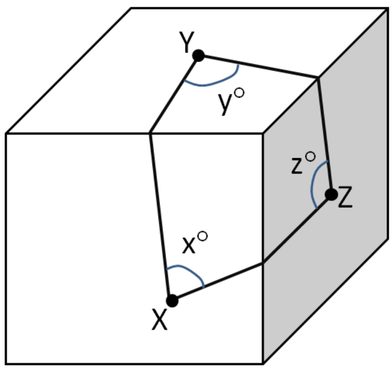


Level: Intermediate Ref No: M88 Puzz Points: 23

[Maclaurin 2008 Q5] An ant lives on the surface of a cuboid which has points X, Y and Z on three adjacent faces.

The ant travels between X, Y and Z along the shortest possible path between each pair of points. The angles $x°$, $y°$ and $z°$ are the angles between the parts of the ant's path, as shown.

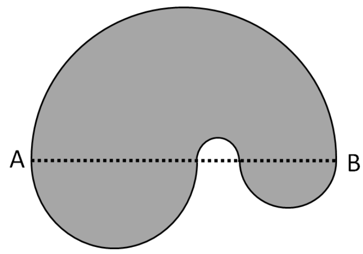
Prove that $x + y + z = 270$.



Level: Intermediate Ref No: M91 Puzz Points: 10

[Cayley 2009 Q2] The boundary of a shaded figure consists of four semicircular arcs whose radii are all different. The centre of each arc lies on the line AB, which is 10cm long.

What is the length of the perimeter of the figure.



Solution: 10π cm

Level: Intermediate Ref No: M93 Puzz Points: 10

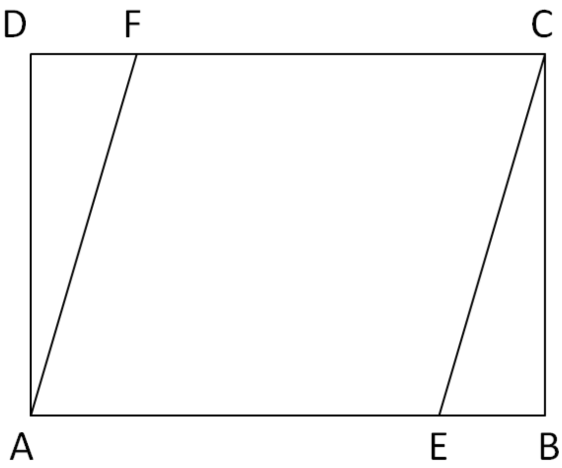
[Cayley 2009 Q4] In a rectangle ABCD, the side AB has length $\sqrt{2}$ and the side AD has length 1. Let the circle with centre B and passing through C meet AB at X.
Find $\angle ADX$ (in degrees).

Solution: $\angle ADX = 22\frac{1}{2}°$

Level: Intermediate   Ref No: M97   Puzz Points: 15

[Hamilton 2009 Q3] In the diagram, ABCD is a rectangle with $AB = 16cm$ and $BC = 12cm$. Points E and F lie on sides AB and CD so that AECF is a rhombus.

What is the length of EF?

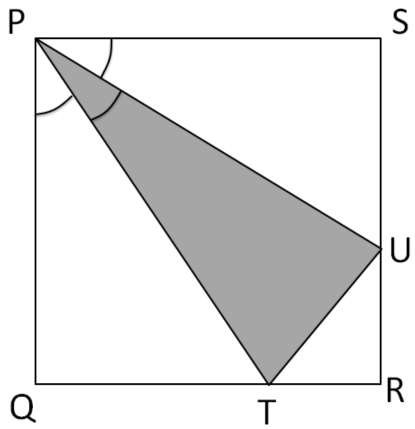


Solution: 15

Level: Intermediate   Ref No: M99   Puzz Points: 18

[Hamilon 2009 Q5] The diagram shows a triangle PTU inscribed in a square PQRS. Each of the marked angles at P is equal to $30°$.

Prove that the area of the triangle PTU is one third of the area of the square PQRS.



Level: Intermediate   Ref No: M104   Puzz Points: 20

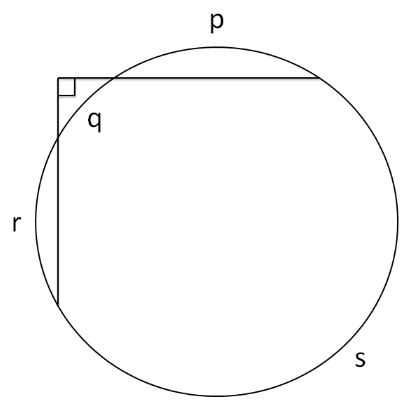
[Maclaurin 2009 Q4] In a trapezium $ABCD$ the sides AB and DC are parallel and $\angle BAD = \angle ABC < 90°$. Point P lies on AB with $\angle CPD = \angle BAD$.

Prove that $PC^2 + PD^2 = AB \times DC$.

Level: Intermediate   Ref No: M106   Puzz Points: 23

[Maclaurin 2009 Q6] In the figure, p, q, r and s are the lengths of the four arcs which together form the circumference of the circle.

Find, in simplified form, an expression for s in terms of p, q and r.

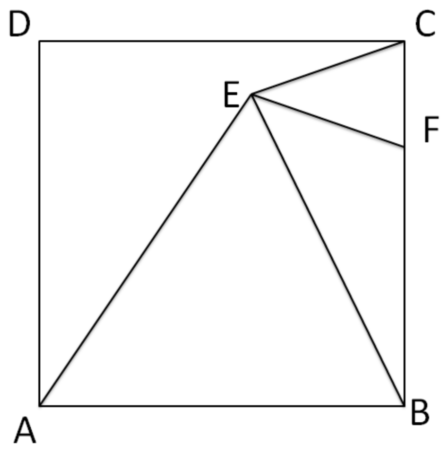


Solution: Length of each arc is proportional to the angle subtended at the centre. With appropriate choice of radius we can make p, q, r, s these angles. Thus $p + q + r + s = 360$. We also have angle sum in a quadrilateral where one angle is q and another $90°$, and using two isosceles triangles, we find $p + 2q + r = 180$. Using these two equations, we obtain $s = p + 3q + r$

Level: Intermediate   Ref No: M108   Puzz Points: 10

[Cayley 2010 Q2] The diagram shows a square ABCD and an equilateral triangle ABE. The point F lies on BC so that EC = EF.

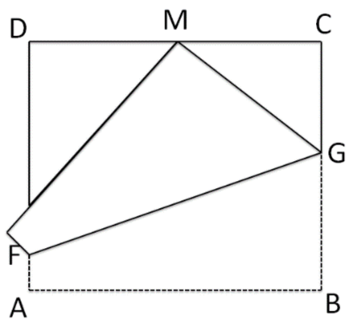
Calculate the angle BEF.



Solution: $45°$

Level: Intermediate Ref No: M111 Puzz Points: 13

[Cayley 2010 Q5] A square sheet of paper ABCD is folded along FG, as shown, so that the corner B is folded onto the midpoint M of CD. Prove that the sides of triangle GCM have lengths in the ratio $3:4:5$.

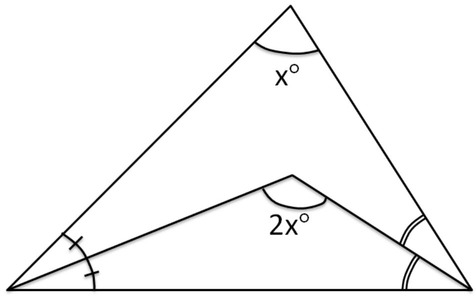


Solution: Let $a = CG$ and $CM = b$. Then using Pythagoras on ΔCGM, $a^2 + b^2 = (2b - a)^2$.
Simplifying we get $a = \frac{3}{4}b$ and $GM = 2b - a = 2b - \frac{3}{4}b = \frac{5}{4}b$.
$\frac{3}{4}b : b : \frac{5}{4}b = 3b : 4b : 5b = 3 : 4 : 5$.

Level: Intermediate Ref No: M113 Puzz Points: 15

[Hamilton 2010 Q2] The diagram shows a triangle and two of its angle bisectors.

What is the value of *x*?

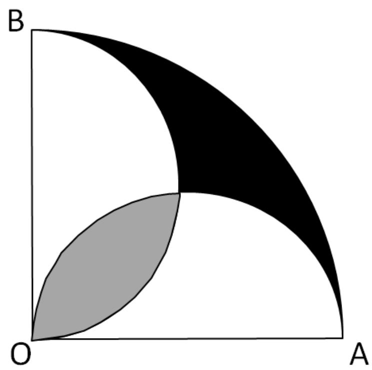


Solution: $x = 60°$

Level: Intermediate Ref No: M115 Puzz Points: 15

[Hamilton 2010 Q4] The diagram shows a quarter-circle with centre O and two semicircular arcs with diameters OA and OB.

Calculate the ratio of the area of the region shaded grey to the area of the region shaded black.

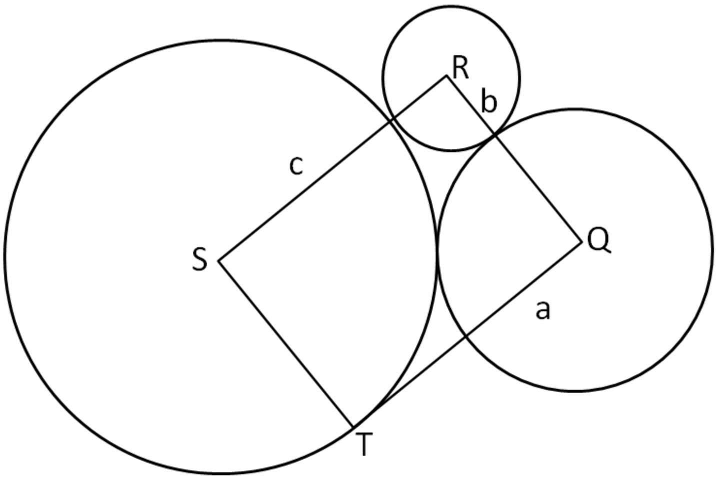


Solution: $1:1$

Level: Intermediate Ref No: M116 Puzz Points: 18

[Hamilton 2010 Q5] The diagram shows three touching circles, whose radii are *a*, *b* and *c*, and whose centres are at the vertices Q, R and S of a rectangle QRST. The fourth vertex *T* of the rectangle lies on the circle with centre *S*.

Find the ratio $a: b: c$.

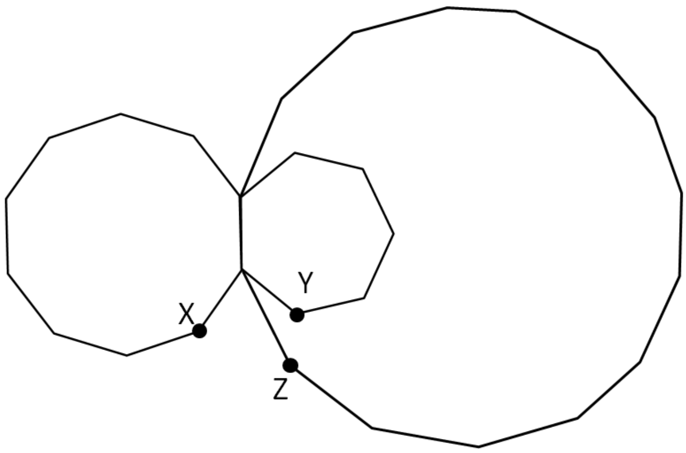


Solution: $a: b: c = 2: 1: 3$

Level: Intermediate   Ref No: M119   Puzz Points: 20

[Maclaurin 2010 Q2] The diagram shows a regular heptagon, a regular decagon and a regular 15-gon with an edge in common.

Find the size of angle *XYZ*.

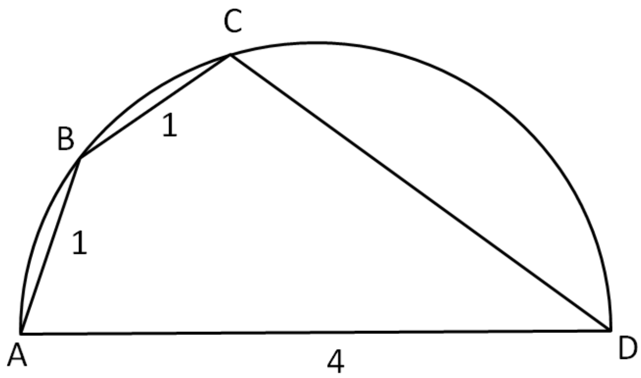


Solution: $\angle XYZ = 30°$

Level: Intermediate   Ref No: M121   Puzz Points: 20

[Maclaurin 2010 Q4] The diameter *AD* of a circle has length 4. The points *B* and *C* lie on the circle, as shown, so that $AB = BC = 1$.

Find the length of CD.

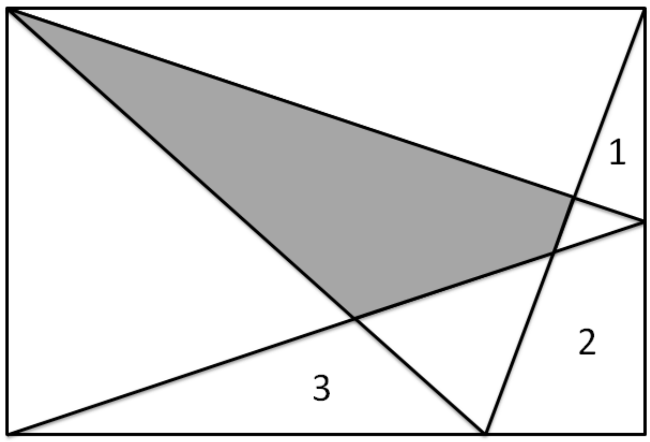


Solution: $CD = \frac{7}{2}$

Level: Intermediate Ref No: M122 Puzz Points: 23

[Maclaurin 2010 Q5] The diagram shows a rectangle divided into eight regions by four straight lines. Three of the regions have areas 1, 2 and 3, as shown.

What is the area of the shaded quadrilateral?



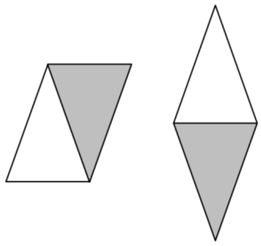
Solution: 6

Level: Intermediate Ref No: M125 Puzz Points: 10

[Cayley 2005 Q2] When two congruent isosceles triangles are joined to form a parallelogram, as shown in the first diagram, the perimeter of the parallelogram is 3cm longer than the perimeter of one of the triangles.

When the same two triangles are joined to form a rhombus, as shown in the second diagram, the perimeter of the rhombus is 7cm longer than the perimeter of one of the triangles.

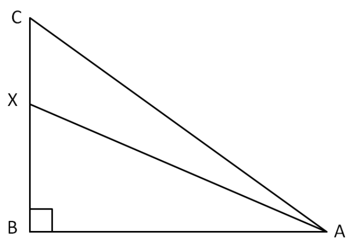
What is the perimeter of one of the triangles?



Solution: 13 cm

Level: Intermediate   Ref No: M126   Puzz Points: 10

[Cayley 2005 Q3] In triangle ABC, angle B is a right angle and X is the point on BC so that $BX:XC = 5:4$. Also, the length of AB is three times the length of CX and the area of triangle CXA is 54cm^2. Calculate the length of the perimeter of triangle CXA.

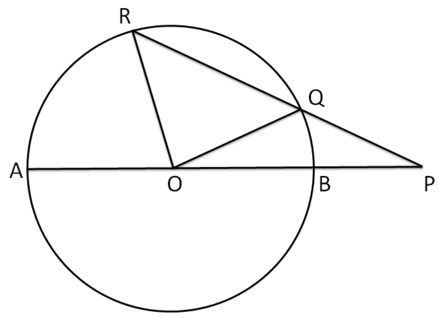


Solution: 48 cm

Level: Intermediate   Ref No: M128   Puzz Points: 13

[Cayley 2005 Q5] In the diagram, O is the centre of the circle and the straight lines AOBP and RQP meet at P. The length of PQ is equal to the radius of the circle. Prove that

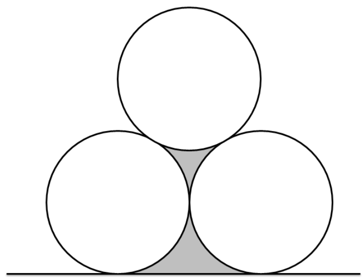
$$\angle AOR = 3 \times \angle BOQ$$



Level: Intermediate   Ref No: M131   Puzz Points: 15

[Hamilton 2005 Q2] The region shown shaded in the diagram is bounded by three touching circles of radius 1 and the tangent to two of the circles.

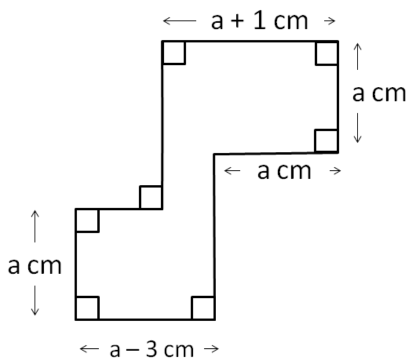
Calculate the perimeter of the shaded region.



Solution: $2\pi + 2$

Level: Intermediate   Ref No: M132   Puzz Points: 15

[Hamilton 2005 Q3] The shape shown in the diagram (not to scale) has a perimeter of length 72cm and an area equal to $147cm^2$. Calculate the value of a.

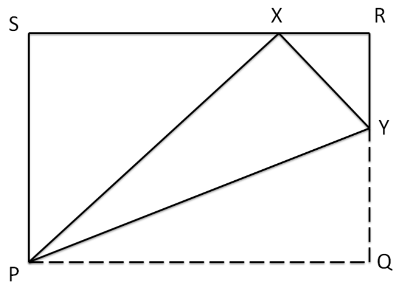


Solution: $a = 9$

Level: Intermediate   Ref No: M134   Puzz Points: 18

[Hamilton 2005 Q5] The rectangle $PQRS$ represents a sheet of A4 paper, which means that $PQ: PS = \sqrt{2}: 1$.

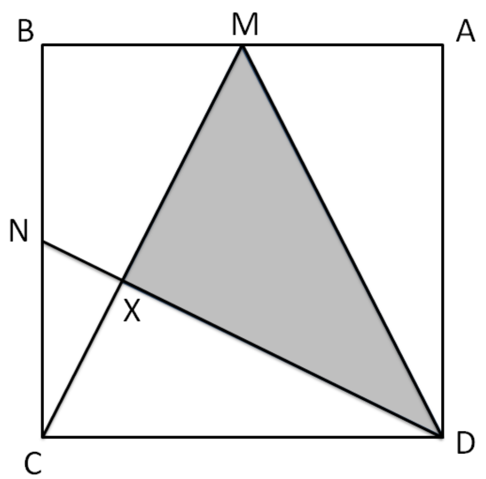
The rectangle is folded, as shown, so that Q comes to a point X on SR and the fold line PY passes though the corner P. Taking the length of PS to be 1 unit, find the lengths of the three sides of the triangle RXY.



Solution: $XY = 2 - \sqrt{2}$ and $RY = RX = \sqrt{2} - 1$

Level: Intermediate Ref No: M140 Puzz Points: 23

[Maclaurin 2005 Q5] In the diagram, X is the point of intersection of lines drawn from the corners C and D of square $ABCD$ to the midpoints M and N of sides AB and BC. Prove that the triangle MXD is right-angled with sides in the ratio $3: 4: 5$.



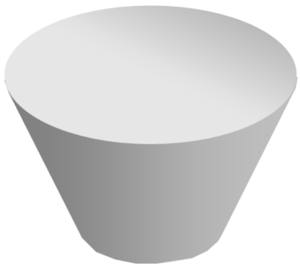
Solution: As ΔXNC and ΔBMC are similar, it follows $\Delta MXD = 90°$. If side of square is 2, then using Pythagoras, $DM = \sqrt{5}$. From similar triangles we find $XN = \frac{\sqrt{5}}{5}$ and thus $DX = \frac{4\sqrt{5}}{5}$. Hence ratio $DX: DM = 4: 5$ and ΔMXD is right-angled, three sides form ratio $3: 4: 5$.

Level: Intermediate Ref No: M141 Puzz Points: 23

[Maclaurin 2005 Q6] A sandcastle has a cylindrical base, on top of which is a second smaller cylinder, with a third even smaller cylinder on top. The three circular cylinders have the same height, and their radii are in the ratio $3: 2: 1$. The height of each cylinder is equal to the radius of the smallest cylinder.

Exactly 24 full buckets of sand were used to construct the sandcastle. The bucket is in the form of a frustum (part of a cone, as shown), whose larger radius equals its perpendicular height, and is twice its smaller radius.

Find the ratio of the total height of the sandcastle to that of the bucket.



Solution: $3: 1$